

# SPLITTING OF THE LANDAU LEVELS BY MAGNETIC PERTURBATIONS AND ANDERSON TRANSITION IN 2D-RANDOM MAGNETIC MEDIA

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ABSTRACT. In this note we consider a Landau Hamiltonian perturbed by a random magnetic potential of Anderson type. For a given number of bands, we prove the existence of both strongly localized states at the edges of the spectrum and dynamical delocalization near the center of the bands in the sense that wave packets travel at least at a given minimum speed. We provide explicit examples of magnetic perturbations that split the Landau levels into full intervals of spectrum.

*Dedicated to the memory of Pierre Duclos (1948 - 2010)*

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## 1. INTRODUCTION

Over the past two decades in the physics literature, a lot of attention has been allocated to random magnetic fields in two dimensions, see e.g. [AHK, BSK, Fu, V] and the references therein. The occurrence of localized states under the sole effect of a random magnetic field has been recurrently predicted by the theory, or computed at the band edges. It has then been a challenging issue to provide evidence of the existence of extended states in a 2D electron gas (2DEG) submitted to random magnetic fields. It is commonly admitted that there are no such extended states in 2DEG with random electric potentials in absence of a constant perpendicular magnetic field (but no mathematical proofs so far!). As far as random magnetic fields are concerned, the issue is harder to settle for subtler effects seem to play a role. For instance, while the computations of [AHK] were favorable to the occurrence of extended states, the ones from [BSK] were indicating their non existence.

In quantum Hall systems, namely a 2DEG submitted to a transverse constant magnetic field, localized states are responsible for the celebrated plateaux of the quantum Hall effect. In the case where the Hall conductance is discontinuous, non trivial transport has been proved to take place in [GKS1] for electric disorder (see also [BeES, GKS2, GKM]). In this note, we provide a similar picture but with magnetic disorder. The random magnetic potential is shown to create both strongly localized states at the edges of the spectrum and dynamical delocalization near the center of the band in the sense that wave packets travel at least at a given minimum speed.

Mathematically, the proof of the occurrence of Anderson localization due to random magnetic potentials only is not an easy task, mainly because of the lack of monotonicity of the eigenvalues as functions of the random variables. Very few preliminary results are available: recently, Ghribi, Hislop, and Klopp [GhHK] proved localization for random magnetic perturbations of a periodic magnetic potential in dimension  $d \geq 2$  (see also [KNNY] for a particular discrete model). They exploit the Wegner estimate obtained in [HK] together with results of Ghribi [Gh] in order to start a multiscale analysis. In [U], Ueki extended [HK] to prove localization for some 2D-magnetic perturbation of the Landau Hamiltonian at the very bottom of the spectrum (below the first Landau level).

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In this note we consider 2D-random magnetic perturbations of the Landau Hamiltonian, and prove a transition between dynamical localization and dynamical delocalization inside an arbitrary number of bands. For our model, the phenomenon is thus similar to that arising for random electric potentials [GKS1, GKS2, GKM]. The proof of localization exploits the Wegner estimate of Hislop and Klopp [HK], revisited by Ghribi, Hislop, and Klopp [GhHK], together with a simple weak disorder argument to start the multiscale analysis, provided some information on the location of the spectrum that we address in a separate argument. Then dynamical localization follows from [GK1, GK2] together with the full set of equivalent properties defining the region of complete localization [GK3, GK4]. Delocalization is proved along the lines of [GKS1]; in particular the Hall conductance is quantized, constant in the region of localization and jumps by one as a Landau level is crossed. To our best knowledge, this is the first 2D-random purely magnetic model for which such a transition has been established mathematically.

If the theory of Anderson localization developed over the past years for a continuum random Schrödinger applies, it remains to prove that it does not lead to an empty result, namely that the interval where states are shown to be localized does intersect the spectrum. Getting detailed enough information about the location of the almost spectrum of the random Hamiltonian is again trickier with non monotonic perturbations of order 1 such as the magnetic ones. In particular, in our setting, the issue reduces to the proof that the Landau level does split as a periodic magnetic perturbation is turned on and that the corresponding spectrum contains an open interval. In [DSS], Dinaburg, Sinai, and Soshnikov considered small periodic electric perturbations of the Landau Hamiltonian, and proved that the low Landau levels split into a set of positive Lebesgue measure. Gruber addressed the same issue in [Gr] but for magnetic perturbations. In this note, we exhibit an explicit family of small periodic magnetic perturbations for which the splitting gives rise to a full interval of spectrum. This is achieved by a direct computation using the translation invariance of our potential in one direction. Such examples are then good enough to be randomized and used as random magnetic fields.

This note is organized as follows. In Section 2 we introduce the model and state the main results. In Section 3 we construct explicit examples for which the low Landau levels split into intervals as the magnetic perturbation is turned on. In Section 4 we prove our main result, Theorem 2.2, while the Appendix, Section 5, contains some technical trace-class estimates used in this proof.

## 2. MAIN RESULTS

Let  $\mathbf{A} = (A_1, A_2) \in L^2_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$  be a magnetic potential. Define the operator  $H(\mathbf{A})$  as the self-adjoint operator generated in  $L^2(\mathbb{R}^2)$  by the closure of the quadratic form

$$\int_{\mathbb{R}^2} |i\nabla u + \mathbf{A}u|^2 dx, \quad u \in C_0^\infty(\mathbb{R}^2).$$

The magnetic field generated by  $\mathbf{A}$  is

$$B(x) := \frac{\partial A_2}{\partial x_1}(x) - \frac{\partial A_1}{\partial x_2}(x), \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

In the case of a constant magnetic field  $B > 0$  introduce the magnetic potential  $\mathbf{A}_0 := (0, Bx_1)$  which generates  $B$ . It is well-known that the spectrum of  $H(\mathbf{A}_0)$  consists of the so-called Landau levels  $(2j-1)B$ ,  $j \in \mathbb{N} := \{1, 2, \dots\}$ , and each Landau level is an eigenvalue of  $H(\mathbf{A}_0)$  of infinite multiplicity. The operator  $H(\mathbf{A}_0)$  is called the Landau Hamiltonian. From now on, the symbol  $B$  stands for the magnetic strength of  $H(\mathbf{A}_0)$ .

Let us introduce the random magnetic potential

$$\mathbf{A}_\omega(x) = \sum_{\gamma \in \mathbb{Z}^2} \omega_\gamma \mathbf{v}_\gamma(x), \tag{2.1}$$

with  $\mathbf{v}_\gamma(x) = (v_1(x - \gamma), v_2(x - \gamma))$ ,  $\gamma \in \mathbb{Z}^2$ ,  $x \in \mathbb{R}^2$ ,  $v_1, v_2$  being two given  $C^1(\mathbb{R}^2, \mathbb{R})$  compactly supported functions, normalized so that  $\|\sum_{\gamma \in \mathbb{Z}^2} \mathbf{v}_\gamma\|_\infty = 1$ ; the random variables  $(\omega_\gamma)_{\gamma \in \mathbb{Z}^2}$  are independent

and identically distributed, supported on  $[-1, 1]$ , with common density

$$\rho_\eta(s) = C_\eta \eta^{-1} \exp(-|s|\eta^{-1}) \chi_{[-1,1]}(s), \quad \eta > 0,$$

and  $C_\eta = (2(1 - e^{-\eta^{-1}}))^{-1}$  the normalization constant (note that  $\frac{1}{2} \leq C_\eta \leq 1$  for  $\eta \in [0, 1]$ ). The support of  $\rho_\eta$  is  $[-1, 1]$  for all  $\eta > 0$ , but as  $\eta$  gets small, the disorder becomes weaker in the sense that for most  $\gamma$  the coupling  $\omega_\gamma$  is small. We may speak of a diluted random model (see [GKS1, GKM] for a similar type of randomness). We denote by

$$H_{B,\lambda,\omega,\eta} := H(\mathbf{A}_0 + \lambda \mathbf{A}_{\omega,\eta}), \quad \lambda > 0,$$

the corresponding magnetic random operator, and will consider small values of the coupling constant  $\lambda$ .

For bounded Borelian functions  $f$ , the maps  $\omega \rightarrow f(H_{B,\lambda,\eta,\omega})$  are measurable. It follows from standard ergodicity arguments that the spectrum is almost surely deterministic as well as its pp, sc and ac components (see e.g. [KM, CFKS, St]). We denote by  $\Sigma_{B,\lambda}$  the almost sure spectrum of  $H_{B,\lambda,\omega,\eta}$  (it does not depend on  $\eta > 0$  since by construction the support of  $\rho_\eta$  is independent of  $\eta > 0$ ). It is easy to see that  $\Sigma_{B,\lambda}$  is contained in a union of intervals  $\mathcal{I}_j(B, \lambda) = [a_j(B, \lambda), b_j(B, \lambda)] \ni (2j-1)B$ ,  $j \in \mathbb{N}$ . Moreover, if<sup>1</sup>  $\mathbb{N} \ni J \lesssim (B\lambda^2)^{-1}$ , then

$$\Sigma_{B,\lambda} \cap (-\infty, (2J-1)B + B] \subset \bigcup_{j=1}^J \mathcal{I}_j(B, \lambda) \subset \bigcup_{j=1}^J [(2j-1)B - C\lambda\sqrt{jB}, (2j-1)B + C\lambda\sqrt{jB}], \quad (2.2)$$

for some constant  $C < \infty$  (see Lemma 4.4 below). As a consequence, for any integer  $J \in \mathbb{N}$ , the first  $J$  intervals  $\mathcal{I}_j(B, \lambda)$ ,  $j = 1, \dots, J$ , are disjoint for  $\lambda$  small enough. More precisely, for any  $B \in (0, \infty)$  there exists  $\lambda_*$  such that for any  $j \leq J$  and any  $\lambda \in [0, \lambda_*)$  we have  $\mathcal{I}_j(B, \lambda) \cap \mathcal{I}_{j+1}(B, \lambda) = \emptyset$ , that is  $b_j(B, \lambda) < a_{j+1}(B, \lambda)$ . We denote by  $\mathbb{G}_j(B, \lambda) = (b_j(B, \lambda); a_{j+1}(B, \lambda))$  the  $j$ -th gap of the spectrum. We say that the couple  $(B, \lambda)$  respects the *disjoint band condition* if we have

$$\mathbb{G}_j(B, \lambda) \neq \emptyset \text{ for any } j \leq J. \quad (2.3)$$

It follows from (2.2) that the disjoint band condition is satisfied if  $\lambda \lesssim \sqrt{B/J}$ .

**Definition 2.1.** *The region of strong dynamical localization for  $H_{B,\lambda,\omega,\eta}$  is denoted by  $\Xi_{(B,\lambda,\eta)}^{SDL} \subset \mathbb{R}$ , and is defined as the set of  $E \in \mathbb{R}$  such that there exists an interval  $I \ni E$  satisfying*

$$\mathbb{E} \left\{ \sup_{t \in \mathbb{R}} \left\| \langle x \rangle^{\frac{p}{2}} e^{-itH_{B,\lambda,\eta,\omega}} \chi_I(H_{B,\lambda,\omega}) \tilde{\chi}_0 \right\|_2^2 \right\} < \infty \quad (2.4)$$

for any  $p > 0$ . Here  $\|\cdot\|_2$  denotes the Hilbert-Schmidt norm,  $\chi_I$  is the characteristic function of  $I$ , and  $\tilde{\chi}_0$  is the characteristic function of the unit square centered at the origin.

Strong dynamical localization is known to characterize the so called region of complete localization, where many localization properties turn out to be equivalent [GK3, GK4]. In particular, this region coincides with the set of energies where the bootstrap multiscale analysis of [GK1] applies.

**Theorem 2.2.** *Fix  $J \in \mathbb{N}$ . Let  $H_{B,\lambda,\omega,\eta}$  be the Hamiltonian described above, satisfying the disjoint band condition (2.3). Then there exists  $\kappa_J > 0$  (depending on  $B$  and  $J$ ) and  $\Lambda = \Lambda(B, J) > 0$ , such that for any  $\lambda \in (0, \Lambda]$  and  $\eta \in (0, c_{B,J}\lambda |\log \lambda|^{-2}]$ , for all  $j = 1, \dots, J$ , the Hamiltonian  $H_{B,\lambda,\eta,\omega}$  exhibits strong dynamical localization, namely for any interval  $I$  satisfying*

$$I \subset \Sigma_{B,\lambda} \cap [a_j(B, \lambda), (2j-1)B - \kappa_J \lambda^2], \quad I \subset \Sigma_{B,\lambda} \cap ((2j-1)B + \kappa_J \lambda^2, b_j(B, \lambda)], \quad (2.5)$$

we have

$$I \subset \Xi_{(B,\lambda,\eta)}^{SDL}. \quad (2.6)$$

Moreover for all  $j = 1, \dots, J$ , there exists a dynamical Anderson transition  $\tilde{E}_j(B, \lambda, \eta) \in \Sigma_{B,\lambda} \cap [(2j-1)B - \kappa_J \lambda^2, (2j-1)B + \kappa_J \lambda^2]$ . More precisely, there exists at least one energy  $\tilde{E}_j(B, \lambda, \eta) \in$

<sup>1</sup>Here and in the sequel we write  $a \lesssim b$  if there exists a constant  $c$  such that  $a \leq cb$ .

$\Sigma_{B,\lambda} \cap [(2j-1)B - \kappa_J \lambda^2, (2j-1)B + \kappa_J \lambda^2]$ , such that for every non-negative  $\mathcal{X} \in C_0^\infty(\mathbb{R})$  with  $\mathcal{X} \equiv 1$  on some open interval containing  $\tilde{E}_j(B, \lambda)$ , and for all  $p > 0$ , we have

$$\frac{1}{T} \int_0^\infty \mathbb{E} \left\{ \left\| \langle x \rangle^{\frac{p}{2}} e^{-itH_{B,\lambda,\eta,\omega}} \mathcal{X}(H_{B,\lambda,\eta,\omega}) \tilde{\chi}_0 \right\|_2^2 \right\} e^{-\frac{t}{T}} dt \geq C_{p,\mathcal{X}} T^{\frac{p}{4}-6}, \quad (2.7)$$

for all  $T \geq 0$  with  $C_{p,\mathcal{X}} > 0$ .

**Remark 2.3.** *A priori, it is not obvious that there exist non empty intervals  $I$  satisfying (2.5). In the proof of Theorem 3.1 below we construct a family of random potentials  $\mathbf{A}_\omega$  and constant magnetic fields  $B$  for which such intervals exist, and provide an estimate of their size.*

**Remark 2.4.** *One may compare Theorem 2.2 to [GKS1, Corollary 2.4] where the only disorder parameter is the dilution coefficient  $\eta$  (see [GKM] as well). In [GKS1, Corollary 2.4], the random potential is an electric one, and, with notations of the present article, the authors consider the case  $\lambda = 1$  and  $\eta$  small. Then localization is proved up to a distance  $\mathcal{O}(\eta |\log \eta|)$  from the Landau levels, while here we get to a distance  $\mathcal{O}(\lambda^2) = o(\eta^{2-\varepsilon})$  for any  $\varepsilon > 0$ , if we take  $\eta = c_{B,J} \lambda |\log \lambda|^{-2}$ . Such a better bound is due to the combined effect of both parameters  $\lambda$  and  $\eta$ .*

### 3. SPLITTING OF THE LANDAU LEVELS

Recall that  $B$  denotes the constant scalar magnetic field generated by the magnetic potential  $\mathbf{A}_0$  of the Landau Hamiltonian  $H(\mathbf{A}_0)$ , and  $\Sigma_{B,\lambda}$  denotes the almost sure spectrum of the operator  $H_{B,\lambda,\omega,\eta} = H(\mathbf{A}_0 + \lambda \mathbf{A}_\omega)$ .

**Theorem 3.1.** *Fix  $J \in \mathbb{N}$ . Then there exists random magnetic potentials  $\mathbf{A}_\omega$  of the form (2.1) with  $\mathbf{v} = (v_1, v_2)$  described explicitly in (3.25), and  $\tilde{\kappa}_J > 0$  and  $\tilde{\lambda}_J > 0$ , such that for any  $B$  in the set  $\mathcal{M}_J = \mathcal{M}_J(\mathbf{A}_\omega) \subseteq (0, \infty)$  described explicitly in (3.23), we have*

$$[(2j-1)B - \tilde{\kappa}_J \lambda, (2j-1)B + \tilde{\kappa}_J \lambda] \subset \Sigma_{B,\lambda}, \quad j = 1, \dots, J, \quad (3.1)$$

provided  $\lambda \in (0, \tilde{\lambda}_J]$ .

**Remark 3.2.** *It follows from (2.2) and Theorem 3.1 that the edges of the almost sure spectrum satisfy*

$$\tilde{\kappa}_J(B) \lambda \leq (a_j(B, \lambda) - (2j-1)B), \quad (b_j(B, \lambda) - (2j-1)B) \leq C \sqrt{jB} \lambda,$$

for all  $j = 1, \dots, J$ , all  $\lambda \in (0, \tilde{\lambda}_J]$ , and all  $\eta > 0$ .

**Remark 3.3.** *The complement of the set  $\mathcal{M}_J$  is always finite. Moreover, Remark 3.8 below provides a simple sufficient condition that  $\mathcal{M}_J = (0, \infty)$ ; informally, this is the generic condition that the periodic function  $a$  appearing in (3.3) has sufficiently many non vanishing Fourier coefficients.*

**Remark 3.4.** *In the construction of the potential  $\mathbf{A}_\omega$  within the proof of Theorem 3.1 we assume that the function  $a$  appearing in (3.3) is given, and describe the set of admissible fields  $B \in (0, \infty)$  for which Theorem 3.1 holds true. Of course, we could start with an arbitrary given  $B$ , and construct afterwards (a family) of 1-periodic  $a \in C^1(\mathbb{R}; \mathbb{R})$  for which Theorem 3.1 is valid.*

*Proof of Theorem 3.1:* The main idea of the proof is to construct an appropriate periodic magnetic potential  $\mathbf{A}_{\text{per}}$  such that for every  $\lambda > 0$  any given Landau level splits into an interval of positive length of the spectrum  $\sigma(H(\mathbf{A}_0 + \lambda \mathbf{A}_{\text{per}}))$  of the operator  $H(\mathbf{A}_0 + \lambda \mathbf{A}_{\text{per}})$ . After that, using ideas of [KM], we show that

$$\sigma(H(\mathbf{A}_0 + \lambda \mathbf{A}_{\text{per}})) \subset \Sigma_{B,\lambda}, \quad (3.2)$$

which implies (3.1). First, we construct  $\mathbf{A}_{\text{per}}$ . Let  $a \in C^1(\mathbb{R}; \mathbb{R})$  be a 1-periodic function whose derivative does not vanish identically. Evidently,  $a$  is bounded on  $\mathbb{R}$ . Set

$$\mathbf{A}_{\text{per}}(x) = (0, a(x_1)), \quad x = (x_1, x_2) \in \mathbb{R}^2. \quad (3.3)$$

Let  $\mathcal{F}$  be the partial Fourier transform with respect to  $x_2$ , i.e.

$$(\mathcal{F}u)(x_1, k) := (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ix_2 k} u(x_1, x_2) dx_2.$$

Then we have

$$\mathcal{F}H(\mathbf{A}_0 + \lambda \mathbf{A}_{\text{per}})\mathcal{F}^* = \int_{\mathbb{R}}^{\oplus} \tilde{h}_{\lambda}(k) dk,$$

where

$$\tilde{h}_{\lambda}(k) := -\frac{d^2}{dx_1^2} + (Bx_1 + \lambda a(x_1) - k)^2, \quad k \in \mathbb{R}, \quad (3.4)$$

is the self-adjoint operator in  $L^2(\mathbb{R})$ , essentially self-adjoint on  $C_0^\infty(\mathbb{R})$ . In (3.4) change the variable  $x_1 = t + k/B$ ,  $t \in \mathbb{R}$ . Then the operator  $\tilde{h}_{\lambda}(k)$  is unitarily equivalent to

$$h_{\lambda}(k) := -\frac{d^2}{dt^2} + (Bt + \lambda a(t + k/B))^2, \quad k \in \mathbb{R}.$$

Note that we have

$$h_{\lambda}(k) = h_0 + v_{\lambda}(k), \quad (3.5)$$

where

$$h_0 := -\frac{d^2}{dt^2} + B^2 t^2 \quad (3.6)$$

and

$$v_{\lambda}(t; k) := 2Bt\lambda a(t + k/B) + \lambda^2 a(t + k/B)^2. \quad (3.7)$$

Since  $B^2 t^2 + v_{\lambda}(t; k) \rightarrow \infty$  as  $t \rightarrow \pm\infty$ , the spectrum of the operator  $h_{\lambda}(k)$  is discrete and simple. Let  $\{E_j(k; \lambda)\}_{j=1}^\infty$  be the increasing sequence of its eigenvalues. Since the operators  $h_{\lambda}(k)$  and  $\tilde{h}_{\lambda}(k)$  are unitarily equivalent, of course,  $\{E_j(k; \lambda)\}_{j=1}^\infty$  is also the increasing sequence of the eigenvalues of  $\tilde{h}_{\lambda}(k)$ .

Applying an appropriate infinite-dimensional version of [K, Theorem 5.16] to the operator  $\tilde{h}_{\lambda}(k)$ , we easily find that the functions  $E_j$ ,  $j \in \mathbb{N}$ , are real analytic functions with respect to  $k$  and  $\lambda$  (see also the first footnote of [K, p. 117]). Moreover,  $E_j(\cdot; \lambda)$ ,  $j \in \mathbb{N}$ , are  $B$ -periodic for any  $\lambda \in \mathbb{R}$ . It is well-known that

$$\sigma(H(\mathbf{A}_0 + \lambda \mathbf{A}_{\text{per}})) = \bigcup_{j \in \mathbb{N}} \bigcup_{k \in [0, B)} \{E_j(k; \lambda)\}. \quad (3.8)$$

Further,

$$E_j(k; 0) = (2j - 1)B, \quad j \in \mathbb{N}, \quad (3.9)$$

i.e. the eigenvalues  $E_j(k; 0)$  coincide with the Landau levels and are independent of  $k \in \mathbb{R}$ . Let  $\varphi_j$ ,  $j \in \mathbb{N}$ , be the real eigenfunctions of the harmonic oscillator  $h_0$  which satisfy  $h_0 \varphi_j = (2j - 1)B \varphi_j$  and  $\int_{\mathbb{R}} \varphi_j(t)^2 dt = 1$ . We recall that

$$\varphi_j(t) = \varphi_j(t; B) = \frac{B^{1/4}}{\sqrt{(j-1)! 2^{j-1} \sqrt{\pi}}} \mathcal{H}_{j-1}(\sqrt{B}t) e^{-Bt^2/2}, \quad j \in \mathbb{N}, \quad t \in \mathbb{R}, \quad (3.10)$$

where

$$\mathcal{H}_q(t) := e^{t^2/2} \left( \frac{d}{dt} - t \right)^q e^{-t^2/2}, \quad q \in \mathbb{Z}_+ := \{0, 1, \dots\}, \quad (3.11)$$

are the Hermite polynomials. Fix  $j \in \mathbb{N}$ . Now, the so-called Feynman-Hellmann formula implies

$$\frac{\partial E_j(k; 0)}{\partial \lambda} = 2B \int_{\mathbb{R}} a(t + k/B) t \varphi_j(t)^2 dt, \quad k \in \mathbb{R}. \quad (3.12)$$

Assume that for some  $k_{\pm} \in [0, B)$  we have

$$\frac{\partial E_j(k_-; 0)}{\partial \lambda} < 0, \quad \frac{\partial E_j(k_+; 0)}{\partial \lambda} > 0. \quad (3.13)$$

Taking into account relations (3.13) and (3.9), as well as the continuity of  $\frac{\partial E_j(k_\pm; \lambda)}{\partial \lambda}$  with respect to  $\lambda$ , we find that there exist  $\varkappa_j > 0$  and  $\lambda_j^* > 0$  such that

$$E_j(k_-; \lambda) - (2j-1)B < -\varkappa_j \lambda, \quad E_j(k_+; \lambda) - (2j-1)B > \varkappa_j \lambda, \quad (3.14)$$

provided that  $\lambda \in (0, \lambda_j^*)$ . Combining (3.12) and (3.13) with (3.14) and (3.8), we obtain the following

**Lemma 3.5.** *Fix  $B > 0$  and  $j \in \mathbb{N}$ . Assume that for some  $k_\pm \in [0, B)$  we have*

$$\int_{\mathbb{R}} a(t + k_-/B) t \varphi_j(t)^2 dt < 0, \quad \int_{\mathbb{R}} a(t + k_+/B) t \varphi_j(t)^2 dt > 0. \quad (3.15)$$

*Then there exist  $\varkappa_j > 0$  and  $\lambda_j^* > 0$  such that*

$$[(2j-1)B - \varkappa_j \lambda, (2j-1)B + \varkappa_j \lambda] \subset \sigma(H(\mathbf{A}_0 + \lambda \mathbf{A}_{\text{per}})),$$

*provided that  $\lambda \in (0, \lambda_j^*)$ .*

Next we establish criteria which guarantee the existence of  $k_\pm$  for which inequalities (3.15) hold true. Expand  $a$  into a Fourier series

$$a(t) = \sum_{l \in \mathbb{Z}} \alpha_l e^{i2\pi l t}, \quad t \in \mathbb{R}, \quad (3.16)$$

with Fourier coefficients

$$\alpha_l = \overline{\alpha_{-l}} := \int_0^1 a(t) e^{-i2\pi l t} dt, \quad l \in \mathbb{Z}.$$

Note that  $a \in C^1(\mathbb{R})$  implies  $\{\alpha_l\}_{l \in \mathbb{Z}} \in \ell^1(\mathbb{Z})$ . Moreover, by the assumption that the derivative of  $a$  does not vanish identically, there exists at least one non vanishing Fourier coefficient  $\alpha_l$  with  $l \in \mathbb{N}$ . Further, we have

$$\begin{aligned} F_j(k) &= F_j(k; B) := \int_{\mathbb{R}} a(t + k/B) t \varphi_j(t; B)^2 dt = \\ &= -2 \sum_{l=1}^{\infty} |\alpha_l| I_j(2\pi l; B) \sin\left(\frac{2\pi k l}{B} + \arg \alpha_l\right), \quad k \in \mathbb{R}, \quad j \in \mathbb{N}, \end{aligned} \quad (3.17)$$

where

$$I_j(s; B) := \int_{\mathbb{R}} \sin(st) t \varphi_j(t; B)^2 dt, \quad s \in \mathbb{R}.$$

Evidently,  $F_j$  is a  $B$ -periodic real analytic function of zero mean value. The existence of  $k_\pm \in [0, B)$  for which inequalities (3.15) hold true, is equivalent to the fact that  $F_j$  does not vanish identically, which on its turn is equivalent to the existence of  $l \in \mathbb{N}$  for which

$$\alpha_l I_j(2\pi l; B) \neq 0. \quad (3.18)$$

**Remark 3.6.** *A condition which guarantees the splitting of the Landau levels into spectral bands of positive length, similar to (3.18), was obtained in [Be, Chapter 4]. Note, however, that in [Be] the Landau Hamiltonian perturbed by a periodic electric potential, was considered.*

Let us now make condition (3.18) more explicit. Fix  $j \in \mathbb{N}$ . Simple calculations yield

$$I_j(s; B) = B^{-1/2} I_j(s B^{-1/2}; 1), \quad s \in \mathbb{R}, \quad (3.19)$$

and

$$I_j(s; 1) = \frac{-1}{(j-1)! 2^{j-1} \sqrt{\pi}} \frac{d}{ds} \int_{\mathbb{R}} e^{ist} e^{-t^2} \mathcal{H}_{j-1}(t)^2 dt, \quad s \in \mathbb{R}. \quad (3.20)$$

Applying [GrRy, Eq. (7.377)] (see also [Be, Lemma 2.2.2]), we get

$$\frac{1}{q! 2^q \sqrt{\pi}} \int_{\mathbb{R}} e^{ist} e^{-t^2} \mathcal{H}_q(t)^2 dt = \mathcal{L}_j(s^2/2) e^{-s^2/4}, \quad s \in \mathbb{R}, \quad q \in \mathbb{Z}_+ \quad (3.21)$$

where

$$\mathcal{L}_q(\xi) := \frac{1}{q!} e^\xi \frac{d^q}{d\xi^q} (\xi^q e^{-\xi}), \quad \xi \in \mathbb{R}, \quad q \in \mathbb{Z}_+,$$

are the Laguerre polynomials. Therefore,

$$I_j(s; 1) = \mathcal{P}_j(s)e^{-s^2/4}, \quad s \in \mathbb{R}, \quad (3.22)$$

where

$$\mathcal{P}_j(s) := \frac{s}{2}(\mathcal{L}_{j-1}(s^2/2) - 2\mathcal{L}'_{j-1}(s^2/2)),$$

and  $\mathcal{L}'_q(\xi) = \frac{d\mathcal{L}_q(\xi)}{d\xi}$ ,  $\xi \in \mathbb{R}$ ,  $q \in \mathbb{Z}_+$ . Obviously,  $\mathcal{P}_j$ ,  $j \in \mathbb{N}$ , is an odd polynomial of degree  $2j-1$ . Hence,  $\mathcal{P}_j$  has at most  $j-1$  distinct positive real roots.

Now for  $a \in C^1(\mathbb{R}; \mathbb{R})$ , recalling (3.16), set

$$\mu_{j,l} := \{s \in (0, \infty) \mid \mathcal{P}_j(2\pi ls) \neq 0\}, \quad l \in \mathbb{N}.$$

Evidently, the complement in  $(0, \infty)$  of  $\mu_{j,l}$  contains no more than  $j-1$  points. Put

$$M_j := \left\{ \beta \in (0, \infty) \mid \beta^{-1/2} \in \bigcup_{l \in \mathbb{N}: \alpha_l \neq 0} \mu_{j,l} \right\}.$$

Relations (3.19) – (3.22) imply that the inclusion  $B \in M_j$  holds if and only if at least one of the coefficients

$$\alpha_l I_j(2\pi l; B) = B^{-1/2} \alpha_l I_j(2\pi l B^{-1/2}; 1), \quad l \in \mathbb{N},$$

in (3.18) does not vanish, i.e. the inclusion  $B \in M_j$  is equivalent to the existence of  $k_{\pm} \in [0, B)$  for which (3.15) is fulfilled.

Moreover,  $B$  is in the complement of  $M_j$ ,  $j \in \mathbb{N}$ , in  $(0, \infty)$ , if and only if  $\mathcal{P}_j(2\pi l B^{-1/2}) = 0$  for each  $l \in \mathbb{N}$  such that  $\alpha_l \neq 0$ . Thus, a simple sufficient (but not necessary) condition that  $M_j = (0, \infty)$ , is that the function  $a$  has at least  $j$  non vanishing Fourier coefficients  $\alpha_l$  with  $l \in \mathbb{N}$ .

Thus, we obtain the following

**Lemma 3.7.** *Fix  $j \in \mathbb{N}$  and let  $a \in C^1(\mathbb{R}; \mathbb{R})$  be given. Then inequalities (3.15) hold for some  $k_{\pm} \in [0, B)$  if and only if  $B \in M_j$ .*

Now we are in position to prove Theorem 3.1. Fix  $J \in \mathbb{N}$  and set

$$\tilde{\lambda}_J := \min_{j=1, \dots, J} \lambda_j^*, \quad \tilde{\kappa}_J = \min_{j=1, \dots, J} \kappa_j,$$

and

$$\mathcal{M}_J := \bigcap_{j=1}^J M_j. \quad (3.23)$$

**Remark 3.8.** *Evidently, the complement of the set  $\mathcal{M}_J$  in  $(0, \infty)$  is always finite, and generically is empty. Similarly to  $M_j$ , a simple sufficient (but not necessary) condition that  $\mathcal{M}_J = (0, \infty)$ , is that the function  $a$  has at least  $J$  non vanishing Fourier coefficients  $\alpha_l$  with  $l \in \mathbb{N}$  (see (3.16)).*

Then Lemmas 3.5 - 3.7 imply that

$$[(2j-1)B - \tilde{\kappa}_J \lambda, (2j-1)B + \tilde{\kappa}_J \lambda] \subset \sigma(H(\mathbf{A}_0 + \lambda A_{\text{per}})), \quad j = 0, \dots, J, \quad (3.24)$$

provided that  $\lambda \in (0, \tilde{\lambda}_J)$ .

Let  $\zeta \in C_0^\infty(\mathbb{R}; \mathbb{R})$  satisfy  $0 \leq \zeta(t) \leq 1$ ,  $\sum_{m \in \mathbb{Z}} \zeta(t-m) = 1$ ,  $t \in \mathbb{R}$ . Define  $\mathbf{A}_\omega$  as in (2.1) with

$$v_1 = 0, \quad v_2(x) = a(x_1)\zeta(x_1)\zeta(x_2), \quad x = (x_1, x_2) \in \mathbb{R}^2. \quad (3.25)$$

Evidently,  $\mathbf{A}_\omega \in C^1(\mathbb{R}^2; \mathbb{R}^2)$ , and

$$\|\mathbf{A}_\omega\|_{L^\infty(\mathbb{R}^2)} + \|\nabla \mathbf{A}_\omega\|_{L^\infty(\mathbb{R}^2)} < \infty,$$

for each realization of the random variables  $(\omega_\gamma)_{\gamma \in \mathbb{Z}^2}$ . Note that if  $\tilde{\omega}$  is the periodic realization of the random variables with  $\tilde{\omega}_\gamma = 1$  for each  $\gamma \in \mathbb{Z}^2$ , then we have  $\mathbf{A}_{\tilde{\omega}} = \mathbf{A}_{\text{per}}$ , the magnetic potential  $\mathbf{A}_{\text{per}}$  being defined in (3.3). Applying a magnetic version of [KM, Theorem 4], we find that (3.2) holds true. Finally, the combination of (3.2) and (3.24) yields (3.1).

## 4. PROOF OF THEOREM 2.2

**4.1. First part: localization.** To prove strong dynamical localization, we perform the bootstrap multiscale analysis of [GK1]. As it is well-known, multiscale analysis in this context requires two main ingredients: a Wegner estimate and an initial scale estimate. We shall play with small enough  $\lambda$ 's and  $\eta$ 's to ensure these two ingredients. Since the spectrum of  $H_{B,\lambda,\eta,\omega}$  shrinks to the Landau levels as  $\lambda \rightarrow 0$ , the constants appearing in the Wegner estimate as well as in the multiscale analysis will grow as  $\lambda \rightarrow 0$ . Since  $\lambda$  will have to be chosen small enough depending on those constants, precise versions of the Wegner estimate and of the initial scale estimate are required.

Let  $\mathcal{Q}_L \subset \mathbb{R}^2$  be the square of side  $L \in 6\mathbb{N}$ , centered at the origin. Let  $\chi_L$  denote the characteristic function of  $\mathcal{Q}_L$ , and  $\Gamma_L$  denote the characteristic function of the set  $\mathcal{Q}_{L-1} \setminus \mathcal{Q}_{L-3}$ . Further, let  $H_{B,\lambda,\eta,\omega,L}$  be the operator  $(-i\nabla - \mathbf{A}_0 - \lambda\mathbf{A}_\omega)^2$  with appropriate boundary conditions, self-adjoint in  $L^2(\mathcal{Q}_L)$ . For  $z \in \mathbb{C} \setminus \sigma(H_{B,\lambda,\eta,\omega,L})$  set  $R_{B,\lambda,\eta,\omega,L}(z) := (H_{B,\lambda,\eta,\omega,L} - z)^{-1}$ .

A Wegner estimate for random magnetic perturbation has been proved in [HK, Theorem 6.1]. Because of the above considerations, our analysis rather relies on the Wegner estimate obtained in [GhHK], and extended to the case of a random magnetic potential in [HK, Theorem 1.2]. More precisely in our context [GhHK, Theorem 4.1] reads:

**Theorem 4.1.** [GhHK] *Let  $E \in ((2j-1)B, (2j+1)B)$ ,  $j \in \mathbb{N}$ , be given and set  $\delta = \text{dist}(E, \sigma(H(\mathbf{A}_0)))$ . Then there exists  $\lambda_0 > 0$  and, for any  $q \in (0, 1)$ , a constant  $Q_q < \infty$  such that for any  $\varepsilon \in (0, \delta/2]$ , any  $\lambda \leq \lambda_0 \min\{1, \delta^{1/2}\}$ , and any  $\eta > 0$ , we have*

$$\mathbb{P} \{ \text{dist}(E, \sigma(H_{B,\lambda,\eta,\omega,L})) \leq \varepsilon \} \leq Q_W \varepsilon^q L^q, \quad (4.1)$$

with  $Q_W = Q_q(\eta\delta)^{-1}$ .

**Remark 4.2.** *The factor  $\eta^{-1}$  in (4.1) comes from the derivative of the probability distribution and [HK, Eq. (3.16)] (see also [U, Theorem 1]).*

For the initial scale estimate, we need the following version of [GK2, Theorem 2.4].

**Theorem 4.3.** [GK2, Theorem 2.4] *Let  $E \in ((2j-1)B, (2j+1)B)$ ,  $j \in \mathbb{N}$ , be given. Set  $\delta := \text{dist}(E, \sigma(H_B))$ . Given a Wegner estimate of the form (4.1), there exist  $C_d, C_{d,q,j} < \infty$ , so that if for  $L \geq C_d \delta^{-\frac{3}{16q}}$  we have*

$$\mathbb{P} \left( C_{d,q,j} B Q_W L^{\frac{16}{3}} \|\Gamma_L R_{B,\lambda,\omega,L}(E) \chi_{L/3}\| < 1 \right) \geq 1 - 2 \cdot 10^{-5}, \quad (4.2)$$

then  $E \in \Xi_{(B,\lambda,\eta)}^{SDL}$ .

In (4.2) we already took into account that the constant  $\gamma_{\mathcal{I}}$  that appears in [GK2, Assumption SLI], is bounded by  $C_d \sqrt{(2j+1)B}$ . This can be seen from [GK3, Theorem A.1], since the magnetic perturbation is relatively bounded with respect to  $H_B$  with relative bound, say,  $\frac{1}{2}$ .

We shall take advantage of the following lemma which is a consequence of the resolvent identity (see [DGR, Lemma 4.1]).

**Lemma 4.4.** *Let  $H(\mathbf{A}_0)$  be the Landau Hamiltonian with constant magnetic field  $B$ . Let  $\mathbf{A} \in \mathcal{C}^1(\mathbb{R}^2)$  be such that  $\|\mathbf{A}\|_\infty \leq K_1 \sqrt{B}$  and  $\|\text{div } \mathbf{A}\|_\infty \leq K_2 B$ . Then there exists a constant  $0 < K_0 < \infty$  such that*

$$\sigma(H(\mathbf{A}_0 + \mathbf{A})) \cap [(2j-1)B - B, (2j-1)B + B] \subset [(2j-1)B - d_j(\mathbf{A}, B), (2j-1)B + d_j(\mathbf{A}, B)], \quad (4.3)$$

for any  $j$  so that where  $d_j(\mathbf{A}, B) < B$ , where  $d_j(\mathbf{A}, B) = K_0 \max(\|\text{div } \mathbf{A}\|_\infty, \|\mathbf{A}\|_\infty \sqrt{(j+1)B})$ .

The same conclusions hold for the finite volume operators, with the same constants, independent of the volume.



By the definition of the probability distribution, given  $\eta \in ]0, 1]$ , we have  $\mathbb{P}(|\omega_0| \leq \alpha) \geq 1 - \exp(-\alpha\eta^{-1})$  (recall that the normalization constant of the probability distribution satisfies  $\frac{1}{2} \leq C_\eta \leq 1$ ). Now, for  $B$  given and  $\lambda \leq \lambda_0$  (given by Theorem 4.1), we note that the spectrum of  $H_{B,\lambda,\eta,\omega,L}$  satisfies

$$\mathbb{P} \left( \sigma(H_{B,\lambda,\eta,\omega,L}) \subset \bigcup_{j=1}^J [B_j - C_j \lambda B^{1/2} \alpha, B_j + C_j \lambda B^{1/2} \alpha] \right) \quad (4.4)$$

$$\geq \mathbb{P}(|\omega_j| \leq \alpha, \forall j \in \Lambda_L) \quad (4.5)$$

$$\geq 1 - \exp(-\alpha\eta^{-1})L^2, \quad (4.6)$$

with  $0 < \alpha < B^{1/2}$ .

Since we are working in spectral gaps, we use the Combes-Thomas estimate of [BCH, Proposition 3.2] (see also the proof of [KlK, Theorem 3.5] based on [BCH, Lemma 3.1]), adapted to a finite volume as in [GK2, Section 3].

Let  $E \in \mathcal{I}_j(B)$  and assume that  $|E - (2j - 1)B| \geq 2\delta$ . We write  $\delta = \kappa_J \lambda^2$ , and choose  $\kappa_J$  so that the condition  $\lambda \leq \lambda_0 \sqrt{\delta}$  in Theorem 4.1 is satisfied, namely  $\kappa_J \geq \lambda_0^{-2}$ . We further use (4.6) with  $\alpha$  such that  $C_J \lambda \sqrt{B} \alpha = \delta$ , that is  $\alpha = C_{B,J} \kappa_J \lambda$ .

We pick  $q \in (0, 1)$  close to 1, and  $L$  such that  $L > \lambda^{-1} \geq C_J \lambda^{-3/(8q)}$  (hence, the assumption  $L \geq C_J \delta^{-3/(16q)}$  in Theorem 4.1 is fulfilled).

Then, using (4.6) and the Combes-Thomas estimate, we conclude that condition (4.2) will be satisfied at the energy  $E$  if

$$\alpha\eta^{-1} \geq C_3 \log L, \quad (4.7)$$

$$C_{J,B} Q_q (\eta\delta)^{-1} L^{\frac{16}{3}} e^{-C_4 \sqrt{\delta} L} < 1, \quad (4.8)$$

where  $C_3 < \infty$  and  $C_4 > 0$ . Recalling that  $\delta = \kappa_J \lambda^2$ , we choose  $L/\log L \geq C_{B,J} \lambda^{-1} \log(\lambda\eta)^{-1}$  so that (4.8) holds, and  $\eta^{-1} \geq C_{B,J} \lambda^{-1} \log L$  so that (4.7) is satisfied. Since for  $\eta$  small enough  $\eta^{-1} \gg \log \log \eta^{-1}$ , these two conditions are compatible, and  $\eta \leq c_{B,J} \lambda |\log \lambda|^{-2}$  is sufficient. As a consequence (4.2) holds for all  $E$  so that  $|E - (2j - 1)B| \geq 2\delta = 2\kappa_J \lambda^2$ , and strong dynamical localization follows.

**4.2. Second part: delocalization.** We finish the proof of Theorem 2.2 following the idea of [GKS1, GKS2] which consists in using the Hall conductance in order to prove the existence of delocalization energies. In [GKS1, GKS2], the authors considered a Landau Hamiltonian perturbed by a random electric perturbation and proved various properties of the Hall conductance, including the fact it is integer valued in delocalization gaps. While both [GKS1] and [GKS2] can be generalized to the case of a random magnetic perturbation, we focus on [GKS1] for it provides a simpler proof which does not require the more involved technology of [GKS2].

Let  $\Lambda$  be the characteristic function of the interval  $[\frac{1}{2}, \infty)$ , and  $\Lambda_j$  be the multiplication operators given by  $\Lambda_j(x) = \Lambda(x_j)$ ,  $j = 1, 2$ . For any orthogonal projection  $P$  such that  $P[[P, \Lambda_1], [P, \Lambda_2]]$  is trace class, we set

$$\Theta(P) := \text{tr}\{P[[P, \Lambda_1], [P, \Lambda_2]]\} = \text{tr}[P\Lambda_1 P, P\Lambda_2 P]. \quad (4.9)$$

Let  $P$  be an orthogonal projection on  $L^2(\mathbb{R}^2)$ , and  $\phi_x$  be a smooth characteristic function of the unit square centered at  $x \in \mathbb{R}^2$ . Assume that we have

$$\|\phi_x P \phi_y\|_2 \leq K_P \langle x \rangle^\kappa \langle y \rangle^\kappa e^{-|x-y|^\zeta} \text{ for any } x, y \in \mathbb{Z}^2, \quad (4.10)$$

with some  $\zeta \in ]0, 1]$ ,  $\kappa > 0$ , and  $K_P < \infty$ . By [GKS1, Lemma 3.1] we have

$$|\Theta(P)| := \|P[[P, \Lambda_1], [P, \Lambda_2]]\|_1 \leq C_{\zeta, \kappa} K_P^2, \quad (4.11)$$

where  $\|\cdot\|_1$  is the trace-class norm. Then the so-called Hall conductance is well-defined, and is given by

$$\sigma_{\text{Hall}}(B, \lambda, \eta, E, \omega) = -2\pi i \Theta(P_{B,\lambda,\eta,E,\omega}). \quad (4.12)$$

Applying the ergodic theorem (see e.g. [GKS1]), we obtain

$$\sigma_{\text{Hall}}(B, \lambda, \eta, E) := \mathbb{E} \sigma_{\text{Hall}, \omega}(B, \lambda, \eta, E, \omega) = \sigma_{\text{Hall}}(B, \lambda, \eta, E, \omega) \text{ for } \mathbb{P} - \text{a.e. } \omega. \quad (4.13)$$

We proceed as in [GKS1] to get the existence of a delocalization energy near each Landau levels using a perturbative argument.

**Lemma 4.5.** *Assume that  $\lambda \lesssim \sqrt{B/J}$  so that the disjoint band condition (2.3) holds. Then  $\sigma_{\text{Hall}}(B, \lambda, \eta, E)$  is constant in each connected component of  $\Xi_{(B, \lambda, \eta)}^{SDL}$ . Moreover for any  $j \leq J$ , we have  $\sigma_{\text{Hall}}(B, \lambda, \eta, E) = j$  whenever  $E \in \Xi_{(B, \lambda, \eta)}^{SDL} \cap (2j-1)B, (2j+1)B[$ .*

*Proof of Lemma 4.5.* That  $\sigma_{\text{Hall}}(B, \lambda, \eta, E)$  is constant in each connected component of  $\Xi_{(B, \lambda, \eta)}^{SDL}$  is a consequence of the strong localization properties of the eigenfunctions that hold in the region of strong dynamical localization. The argument follows from Lemma 3.1 and Lemma 3.2 of [GKS1] which are general results, independent of the particular form of the random perturbation.

The proof of the second assertion is standard and consists in starting with the zero disorder situation and a energy  $E$  in the middle of a given gap  $\mathbb{G}_j(B, 0)$ , where the Hall conductance  $\sigma_{\text{Hall}}(B, \lambda, \eta, E)$  is known to be equal to  $j$  (e.g. [AvSS, BeES]); then increase the disorder parameter  $\lambda$  keeping  $E$  in  $\mathbb{G}_j(B, \lambda)$  and show that the conductance remains constant; at last, use the fact that the Hall conductance  $\sigma_{\text{Hall}}(B, \lambda, \eta, E)$  is constant when moving the energy  $E$  inside a region of dynamical localization.

The first step, namely increasing the disorder, is a perturbative argument which is performed here for magnetic perturbations along the lines of proof of [GKS1, Lemma 3.3].

Pick  $E = 2jB$ , the middle of the gap  $\mathbb{G}_j(B, 0)$ . Since the gap remains open for sufficiently small  $\lambda \geq 0$ , we can write  $P_\lambda = P_{B, \lambda, \eta, \omega, E}$  as an appropriate Riesz projection, apply the Combes-Thomas theory, and obtain the estimate

$$\|\phi_x P_\lambda \phi_y\|_2 \leq K_1 e^{-K_1 |x-y|} \text{ for all } x, y \in \mathbb{Z}^2 \text{ and } \lambda \in I,$$

with some  $K_1 > 0$  depending on  $\eta$  (cf. [GKS1, Eq. (3.16)]). In particular, (4.10) holds true. Suppose now that the perturbation  $\mathbf{A}_\omega$  has a compact support. By Lemma 5.1 the operator  $\mathcal{Q}_{\lambda, \lambda'} := P_\lambda - P_0 - (P_{\lambda'} - P_0)$  is trace-class for all  $\lambda, \lambda' \in I$ . Using the second form of  $\Theta(P_\lambda)$  in (4.9) and expanding the difference  $\Theta(P_\lambda) - \Theta(P_{\lambda'})$  in four terms with  $P_\lambda = P_{\lambda'} + \mathcal{Q}_{\lambda, \lambda'}$  as in [GKS1, Eq. (3.35)] yields  $\Theta(P_\lambda) = \Theta(P_{\lambda'})$ .

Next, we use an approximation argument considering  $\omega^L, \omega^{>L}$  given by  $\omega_i^L = \omega_i$  if  $|i| \leq L$  and  $\omega_i^L = 0$  otherwise, and  $\omega_i^{>L} = \omega_i - \omega_i^L$  for any  $L > 0$ . With the obvious notations, we set  $\mathcal{Q}_{\lambda, >L} := P_\lambda - P_{\lambda, L}$  (cf. [GKS1, Eq. (3.36)]). Using an appropriate Combes-Thomas estimate (see e.g. [CG, Lemma A.3]) and a smooth partition of unity  $\{\phi_x\}_{x \in \mathbb{Z}^2}$ , we find that

$$\|\phi_x \mathcal{Q}_{\lambda, >L} \phi_y\| \leq C e^{-C(|x-y| + \max\{L-|x|, 0\} + \max\{L-|y|, 0\})}. \quad (4.14)$$

Putting together (4.14) and (5.13), we obtain the estimate

$$\|\phi_x \mathcal{Q}_{\lambda, >L} \phi_y\|_2 \leq \|\phi_x \mathcal{Q}_{\lambda, >L} \phi_y\|_1^{\frac{1}{2}} \|\phi_x \mathcal{Q}_{\lambda, >L} \phi_y\|_2^{\frac{1}{2}} \quad (4.15)$$

$$\leq C' e^{-C'(|x-y| + \max\{L-|x|, 0\} + \max\{L-|y|, 0\})}, \quad (4.16)$$

for all  $x, y \in \mathbb{Z}^2$  and  $L > 0$ . Combining the fact that for any orthogonal projections  $P_\alpha, P_\beta, P_\gamma$  we have

$$\|P_\alpha[[P_\beta, \Lambda_1], [P_\gamma, \Lambda_2]]\|_1 \leq \sum_{x, y, z \in \mathbb{Z}^2} \|\phi_x [P_\beta, \Lambda_1] \phi_y\|_2 \|\phi_y [P_\gamma, \Lambda_1] \phi_z\|_2,$$

with (4.15), and the dominated convergence theorem, we get that  $\Theta(P_\lambda) - \Theta(P_{\lambda, L}) \xrightarrow{L \rightarrow \infty} 0$ . This ends the proof of the lemma.  $\square$

We now finish the proof of the Theorem 2.2. Let us fix the couple  $(B, \lambda)$  so that the disjoint band condition (2.3) is valid. Pick  $j \leq J$ . By virtue of Lemma 4.5, it is not possible that  $\mathcal{I}_j(B, \lambda) \subset \Xi_{(B, \lambda, \eta)}^{SDL}$ .

As a consequence, there exists at least one energy  $\tilde{E}_j(B, \lambda, \eta) \in \mathcal{I}_j(B, \lambda)$  such that  $\tilde{E}_j(B, \lambda, \eta) \notin \Xi_{(B, \lambda, \eta)}^{SDL}$ . Next, because of (2.5) and (2.6), we have  $\tilde{E}_j(B, \lambda, \eta) \in [(2j-1)B - \kappa_J \lambda^2, (2j-1)B + \kappa_J \lambda^2]$ . Finally, to get the dynamical lower bound (2.7) near  $\tilde{E}_j(B, \lambda, \eta)$  we apply [GK3, Theorem 2.11]. We can indeed readily apply [GK3] to magnetic random perturbations: the bounds from [GK3, Appendix A] are valid within our context since  $H_{B, \lambda, \omega, \eta} - H(\mathbf{A}_0)$  is relatively  $H(\mathbf{A}_0)$ -bounded with relative bound  $< 1$ , and the proof of [GK3, Theorem 2.11] itself works as well for magnetic perturbations.

## 5. APPENDIX: TRACE ESTIMATES

Let  $\mathbf{a} = (a_1, a_2) \in L_{\text{loc}}^2(\mathbb{R}^2; \mathbb{R}^2)$ . Introduce the self-adjoint operator  $H_\lambda := (-i\nabla - \mathbf{A}_0 - \lambda \mathbf{a})^2$ ,  $\lambda \geq 0$ , where, as earlier, the magnetic potential  $\mathbf{A}_0$  generates a constant magnetic field  $B > 0$ . Denote by  $P_{\lambda, E}$  the spectral projection of the operator  $H_\lambda$  associated with the interval  $(-\infty, E)$ ,  $E \in \mathbb{R}$ . We will say that  $\mathcal{E} \in \mathbb{R}$  is in a spectral gap of the family  $H_\lambda$ ,  $\lambda \in [0, \lambda_0]$ , with some  $\lambda_0 > 0$ , if there exist closed disjoint intervals  $\mathcal{J}_-$  and  $\mathcal{J}_+$  such that

$$(-\infty, \mathcal{E}) \cap \bigcup_{\lambda \in [0, \lambda_0]} \sigma(H_\lambda) \subseteq \mathcal{J}_-, \quad (\mathcal{E}, \infty) \cap \bigcup_{\lambda \in [0, \lambda_0]} \sigma(H_\lambda) \subseteq \mathcal{J}_+.$$

**Lemma 5.1.** *Assume that  $\mathbf{a} \in C^1(\mathbb{R}^2; \mathbb{R}^2)$  has a compact support. Let  $\mathcal{E} \in \mathbb{R}$  is in a spectral gap of the family  $H_\lambda$ ,  $\lambda \in [0, \lambda_0]$  with some  $\lambda_0 > 0$ . Then the operator  $P_{\lambda, \mathcal{E}} - P_{0, \mathcal{E}}$  is trace-class for all  $\lambda \in [0, \lambda_0]$ .*

*Proof.* Evidently, there exists a bounded contour  $\Gamma$  such that  $\mathcal{J}_-$  is contained in its interior,  $\mathcal{J}_+$  is contained in its exterior, and there exists  $s > 0$  such that  $\text{dist}(\Gamma, \sigma(H_\lambda)) > s$  for every  $\lambda \in [0, \lambda_0]$ . For  $z \in \mathbb{C} \setminus \sigma(H_\lambda)$  write  $R_\lambda(z) = (H_\lambda - z)^{-1}$ . Then we have

$$P_{\lambda, \mathcal{E}} - P_{0, \mathcal{E}} = \frac{1}{2\pi i} \int_\Gamma R_\lambda(z) \mathcal{W} R_0(z) dz \quad (5.1)$$

where

$$\mathcal{W} = \mathcal{W}_\lambda := H_\lambda - H_0 = 2\lambda \mathbf{a} \cdot (-i\nabla - \mathbf{A}_0) + i\lambda \text{div } \mathbf{a} + \lambda^2 |\mathbf{a}|^2.$$

Let  $\zeta_0 \in C_0^\infty(\mathbb{R}^2; \mathbb{R})$  be a cut-off function, equal to one on the support of  $\mathbf{a}$ . Then we have

$$R_\lambda(z) \mathcal{W} R_0(z) = \zeta_0 R_\lambda(z) \mathcal{W} R_0(z) \zeta_0 + R_\lambda(z) \zeta_1 S_\lambda R_\lambda(z) \mathcal{W} \zeta_0 R_0(z) + \zeta_0 R_\lambda(z) \mathcal{W} R_0(z) S_0 \zeta_1 R_0(z) \quad (5.2)$$

where

$$S_\lambda = [H_\lambda(z), \zeta_0] := 2i\nabla \zeta_0 \cdot (-i\nabla - \mathbf{A}_0 - \lambda \mathbf{a}) - \Delta \zeta_0, \quad \lambda \geq 0,$$

and  $\zeta_1 \in C_0^\infty(\mathbb{R}^2; \mathbb{R})$  is a cut-off function, equal to one on the support of  $\zeta_0$ . Obviously,

$$\left\| \int_\Gamma R_\lambda(z) \zeta_1 S_\lambda R_\lambda(z) \mathcal{W} \zeta_0 R_0(z) dz \right\|_1 \leq |\Gamma| \sup_{z \in \Gamma} (\|R_\lambda(z) \zeta_1\|_2 \|S_\lambda R_\lambda(z) \mathcal{W}\| \|\zeta_0 R_0(z)\|_2), \quad (5.3)$$

$$\left\| \int_\Gamma \zeta_0 R_\lambda(z) \mathcal{W} R_0(z) S_0 \zeta_1 R_0(z) dz \right\|_1 \leq |\Gamma| \sup_{z \in \Gamma} (\|\zeta_0 R_\lambda(z)\|_2 \|\mathcal{W} R_0(z) S_0\| \|\zeta_1 R_0(z)\|_2), \quad (5.4)$$

where  $\|\cdot\|_1$  denotes the trace-class norm, and  $|\Gamma|$  is the length of  $\Gamma$ . Applying the Hilbert-Schmidt diamagnetic inequality (see e.g. [Si, Theorem 2.13]), to the operators  $\zeta_j R_\lambda(-1)$ , we find that

$$\|\zeta_j R_\lambda(z)\|_2^2 = \|R_\lambda(z) \zeta_j\|_2^2 \leq \frac{c_0^2}{(2\pi)^2} \|\zeta_j\|_{L^2(\mathbb{R}^2)}^2 \int_{\mathbb{R}^2} \frac{d\xi}{(|\xi|^2 + 1)^2}, \quad j = 0, 1, \quad z \in \Gamma, \quad \lambda \in [0, \lambda_0],$$

with

$$c_0 := \sup_{z \in \Gamma} \sup_{\lambda \in [0, \lambda_0]} \sup_{E \in \sigma(H_\lambda)} \frac{E + 1}{|E - z|}.$$

Similarly,

$$\sup_{z \in \Gamma} \|S_\lambda R_\lambda(z) \mathcal{W}\| < \infty, \quad \sup_{z \in \Gamma} \|\mathcal{W} R_0(z) S_0\| < \infty.$$

Further, set  $B_j := B(2j - 1)$ ,  $j \in \mathbb{N}$ , and write  $R_0(z) = \sum_{j \in \mathbb{N}} (B_j - z)^{-1} \Pi_j$  where  $\Pi_j$  is the orthogonal projection onto  $\text{Ker}(H_0 - B_j)$ . Put

$$R_\lambda^-(z) := \int_{(-\infty, \mathcal{E})} (E - z)^{-1} d_E P_{\lambda, E}, \quad z \in \mathbb{C} \setminus (-\infty, \mathcal{E}),$$

$$R_\lambda^+(z) := \int_{(\mathcal{E}, \infty)} (E - z)^{-1} d_E P_{\lambda, E}, \quad z \in \mathbb{C} \setminus (\mathcal{E}, \infty).$$

By the Cauchy theorem,

$$\frac{1}{2\pi i} \int_\Gamma \zeta_0 R_\lambda(z) \mathcal{W} R_0(z) \zeta_0 dz = \sum_{j \in \mathbb{N} : B_j \in \mathcal{J}_+} \zeta_0 R_\lambda^-(B_j) \mathcal{W} \zeta_0 \Pi_j \zeta_0 - \sum_{j \in \mathbb{N} : B_j \in \mathcal{J}_-} \zeta_0 R_\lambda^+(B_j) \mathcal{W} \zeta_0 \Pi_j \zeta_0. \quad (5.5)$$

Let us estimate the trace-class norm of the first (infinite) sum at the r.h.s. of (5.5). For each  $j \in \mathbb{N}$  such that  $B_j \in \mathcal{J}_+$  we have

$$\begin{aligned} & \zeta_0 R_\lambda^-(B_j) \mathcal{W} \zeta_0 \Pi_j \zeta_0 = \\ & \zeta_0 R_\lambda^-(B_j) (H_\lambda + 1)^2 R_\lambda(-1) \zeta_0 R_\lambda(-1) \mathcal{W} \zeta_0 \Pi_j \zeta_0 + \zeta_0 R_\lambda^-(B_j) (H_\lambda + 1)^2 R_\lambda(-1) \zeta_1 S_\lambda R_\lambda(-1)^2 \mathcal{W} \zeta_0 \Pi_j \zeta_0. \end{aligned}$$

Therefore,

$$\begin{aligned} & \left\| \sum_{j \in \mathbb{N} : B_j \in \mathcal{J}_+} \zeta_0 R_\lambda^-(B_j) \mathcal{W} \zeta_0 \Pi_j \zeta_0 \right\|_1 \leq \\ & (\|R_\lambda(-1) \zeta_0\|_2 \|R_\lambda(-1) \mathcal{W}\| + \|R_\lambda(-1) \zeta_1\|_2 \|S_\lambda R_\lambda(-1)^2 \mathcal{W}\|) \sum_{j \in \mathbb{N} : B_j \in \mathcal{J}_+} \|\zeta_0 R_\lambda^-(B_j) (H_\lambda + 1)^2\| \|\zeta_0 \Pi_j \zeta_0\|_2. \end{aligned} \quad (5.6)$$

By the spectral theorem,

$$\|\zeta_0 R_\lambda^-(B_j) (H_\lambda + 1)^2\| \leq \|\zeta_0\|_{L^\infty(\mathbb{R}^2)} \sup_{E \in \mathcal{J}_-} \frac{(E + 1)^2}{|E - B_j|} \leq c_1 j^{-1} \quad (5.7)$$

where  $c_1$  is independent of  $j$ . Next, [KoPu, Lemma 3.1] implies

$$\|\zeta_0 \Pi_j \zeta_0\|_2 \leq c_2 j^{-1/4} \quad (5.8)$$

with  $c_2$  independent of  $j$ . Putting together (5.6), (5.7), and (5.8), we conclude that there exists  $c_3$  such that

$$\left\| \sum_{j \in \mathbb{N} : B_j \in \mathcal{J}_+} \zeta_0 R_\lambda^-(B_j) \mathcal{W} \zeta_0 \Pi_j \zeta_0 \right\|_1 \leq c_3 \sum_{j \in \mathbb{N}} j^{-5/4} < \infty. \quad (5.9)$$

Finally, we estimate the trace-class norm of the second (finite) sum at the r.h.s. of (5.5). We have

$$\left\| \sum_{j \in \mathbb{N} : B_j \in \mathcal{J}_-} \zeta_0 R_\lambda^+(B_j) \mathcal{W} \zeta_0 \Pi_j \zeta_0 \right\|_1 \leq \|\zeta_0\|_{L^\infty(\mathbb{R}^2)} \sum_{j \in \mathbb{N} : B_j \in \mathcal{J}_-} \|R_\lambda^+(B_j) \mathcal{W}\| \|\zeta_0 \Pi_j \zeta_0\|_1.$$

Moreover,

$$\|\zeta_0 \Pi_j \zeta_0\|_1 = \|\Pi_j \zeta_0\|_2^2 = \frac{B}{2\pi} \|\zeta_0\|_{L^2(\mathbb{R}^2)}^2, \quad j \in \mathbb{N}, \quad (5.10)$$

(see e.g. [FR, Lemma 3.1]). Since the number of the Landau levels  $B_j$  lying on  $\mathcal{J}_-$  is finite, and the operators  $R_\lambda^+(B_j) \mathcal{W}$  are bounded provided that  $B_j \in \mathcal{J}_-$ , we get

$$\left\| \sum_{j \in \mathbb{N} : B_j \in \mathcal{J}_-} \zeta_0 R_\lambda^+(B_j) \mathcal{W} \zeta_0 \Pi_j \zeta_0 \right\|_1 < \infty. \quad (5.11)$$

Combining (5.1) – (5.3), (5.4), (5.9), and (5.11), we find that the operator  $P_{\lambda, \mathcal{E}} - P_{0, \mathcal{E}}$  is trace-class.  $\square$

**Lemma 5.2.** *Let  $\mathbf{a} \in C^1(\mathbb{R}^2; \mathbb{R}^2)$  with*

$$\|\mathbf{a}\|_{L^\infty(\mathbb{R}^2)} + \|\nabla \mathbf{a}\|_{L^\infty(\mathbb{R}^2)} \leq K, \quad (5.12)$$

*with some  $K < \infty$ . Suppose that  $\mathcal{E} \in \mathbb{R}$  is in a spectral gap of the family  $H_\lambda$ ,  $\lambda \in [0, \lambda_0]$  with some  $\lambda_0 > 0$ . Let  $\phi_x$  be a smooth characteristic function of the unit square centered at  $x \in \mathbb{R}^2$ . Then for any  $x, y \in \mathbb{R}^2$  the operator  $\phi_x P_{\lambda, \mathcal{E}} \phi_y$  is trace-class, and we have*

$$\|\phi_x P_{\lambda, \mathcal{E}} \phi_y\|_1 \leq C, \quad (5.13)$$

*with  $C = C(K)$  independent of  $x, y \in \mathbb{R}^2$ ,  $\lambda \in [0, \lambda_0]$ , and of  $\mathbf{a}$  satisfying (5.12).*

The proof of the proposition is quite similar to the previous one so that we omit the details, and only note that the analogues of the bounds obtained in the proof of Lemma 5.1 remain uniform with respect to  $\mathbf{a}$  satisfying (5.12), and the norms  $\|\zeta_j\|_{L^p(\mathbb{R}^2)}$  of the cut-off functions, as well as the constant  $c_2$  in (5.8), are invariant under translations of the supports of  $\zeta_j$ .

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